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CIRCLES ORTHOGONAL TO A GIVEN SPHERE

By C. L. E. Moore

In his dissertation Dr. Forbes* has discussed analytically the geometry of circles orthogonal to a given sphere and by means of coordinates has established the relation between this geometry and the ordinary geometry of the straight line. It is the object of this paper to develop the geometry of circles orthogonal to a given sphere by means of a transformation which transforms straight lines into circles orthogonal to a fixed sphere. By this method we are enabled to obtain the properties of the surface of singularities of the quadratic complex which we were unable to do by means of the analysis. The transformation used has been partially discussed by Laguerre† and Darboux‡

1. The Transformation. In order to define this transformation we take a sphere of reference S with center at O. Every plane P determines with S a pencil of spheres which contains two point spheres. Calling the centers of these point spheres p and p', *the transformation we wish to consider is the one which carries the plane P over into the points p, p'.* We shall call p and p' the representative point pair of the plane P. It is well known that these two points are related to each other by inversion with respect to S.

* C. S. Forbes, Geometry of circles orthogonal to a given sphere, New York, 1904.

† *Œuvres*, vol. II, p. 54.

‡ *Sur une classe remarquable des courbes, etc.*, p. 123.

We begin by establishing the following property of this transformation.

If the plane P is revolved about a line l the points p, p' will generate a circle C orthogonal to S whose center lies on l and whose plane passes through O and is perpendicular to l.

Since the centers of the spheres of the pencil formed by S and P lie on a line passing through O and perpendicular to P it follows that as P is revolved about l the centers of the point spheres p, p' will lie in a plane passing through O and perpendicular to l. This plane will cut S in a great circle C_1 and will cut the book of planes having l for axis in a pencil of lines V. It is evident that the points p, p' are the centers of the point circles of the pencils determined by C_1 and the line of V which corresponds to P. With the vertex of V as center draw a circle C orthogonal to C_1. Then since C is orthogonal to C_1 and to each line of the pencil V it is orthogonal to the pencil determined by C_1 and any line of V, and hence contains the points p and p' (by familiar properties of a pencil of circles). Hence the theorem is established. Conversely all the points of C can be obtained as the centers of the point circles of one of the above pencils. For it is well known from the property of orthogonal circles that a line drawn through O will cut C in points p, p' related by inversion with respect to C_1 and p and p' are the point circles of the pencil determined by C_1 and the line bisecting the chord joining pp'.

In view of the theorem just proved the line l will be said to be transformed into the circle C.

From what has just been said it follows at once that:

If the plane P is pivoted about a point O' the points p, p' will generate a sphere S' orthogonal to S and having O' for center.

In this sense the point O' will be said to be transformed into the sphere S'.

It is well known that the common circle C of a pencil of spheres orthogonal to S is itself orthogonal to S. From the theorem stated above the points of a line l are transformed into a pencil of spheres orthogonal to S. The common circle C of the pencil is orthogonal to S; its center lies on l and its plane passes through O (since this plane belongs to the pencil of spheres orthogonal to S); it is therefore the same circle which was obtained by transforming the planes containing l. The circle C may therefore be considered either as the locus of the point pairs p, p' obtained by transforming the planes containing l or as the envelope of the spheres S' obtained by transforming the points of l.

The results of the transformation may be tabulated as follows.

Original element	*Transformed element*
A plane.	A pair of points related to each other by inversion with respect to S.
A line.	A circle orthogonal to S having its center on the line and lying in a plane perpendicular to the line and passing through O.
A point.	A sphere orthogonal to S and having the point for center.
Points on a line.	Spheres having a circle orthogonal to S in common.
Planes having a line in common.	Point pairs lying on a circle orthogonal to S.
A plane and a point on it.	A point pair lying on a sphere orthogonal to S.
A plane and a line in it.	A point pair lying on a circle orthogonal to S.
A line and a point on it.	A circle orthogonal to S lying on a sphere orthogonal to S.

The exceptional elements of the transformation are minimum planes, the plane at infinity, points in the plane at infinity, and lines in the plane at infinity. The result of transforming these elements is given in the table below. These facts will be evident to those who are accustomed to reasoning about minimum planes, the plane at infinity, etc., and for those who are not so accustomed the proof will be given in the appendix.

Original element	*Transformed element*
A minimum plane (a plane which cuts the circle at infinity in a single point).	The point where the plane touches the circle at infinity together with a definite point on the line joining O to this point.
The plane at infinity.	The point O together with any point whatever in the plane at infinity.

A point in the plane at infinity.	A plane passing through O together with the plane at infinity itself.
A line in the plane at infinity.	A line passing through O together with a line in the plane at infinity.

From the properties of the transformation it follows at once that a line l which is tangent to S is transformed into the point circle whose center is at the point of tangency and whose plane is perpendicular to l and passes through O.

Any configuration of lines is transformed into the corresponding configuration of circles orthogonal to S. Lines which intersect are transformed into circles which intersect in the representative point pair of their common plane. Thus we see that circles orthogonal to S which intersect in one point intersect in a second point related to the first by inversion with respect to S. The ∞^2 lines in a plane are transformed into the ∞^2 circles orthogonal to S which intersect in the representative point pair of the plane. The ∞^2 lines passing through a point O' are transformed into the ∞^2 circles orthogonal to S which lie on S', the sphere into which O' is transformed. It is easily seen that the centers of these circles lie on a second sphere which has OO' for diameter.

If we apply the above transformation to the points of a non-developable surface A we obtain a surface enveloped by spheres orthogonal to S and having their centers on A. In order to find the points of contact of any enveloping sphere R with the transformed surface it is necessary to find the limiting points of intersection of enveloping spheres as they approach coincidence with R. The points of A which lie in a plane are transformed into spheres which have the representative point pair of the plane in common. Now as the plane approaches coincidence with the plane tangent to A at the center of R the points of intersection of the spheres approach coincidence with the representative point pair of the tangent plane. From which we see that we obtain the points of contact of any enveloping sphere R by transforming the plane tangent to A at the center of R. Therefore:

The surfaces obtained by transforming the points of A and the planes tangent to A are identical.

It is evident that the surfaces obtained by transforming the points of a curve and the tangent lines to the curve are identical.

We will now consider the cases when the surface A is a scroll (ordinary ruled surface) or a developable. Since any plane containing a generator of a

scroll is tangent to the scroll the surfaces obtained by transforming the generators and by transforming the tangent planes are identical. The generators of the scroll are transformed into the circle generators of a circled surface. This surface cannot be generated as the envelope of a one parameter family of spheres, since it cannot be obtained by transforming a one parameter family of points. This is expressed by saying the surface is non-annular.

If the surface A is a developable surface and is considered as generated by the tangent lines to a twisted curve it is transformed into a circled surface which can be generated as the envelope of a one parameter family of spheres; for the same surface can be obtained by transforming the points of the twisted curve. This surface is an annular surface. If the developable is considered as the envelope of the osculating planes to a twisted curve it is transformed into a second twisted curve (since it is obtained by transforming a one parameter family of planes), which is the locus of the limiting points of intersection of the circles of the annular surface above. For if we transform any three points of the original curve, the points of intersection of the circles common to the three spheres will be the representative point pair of the plane of the three points. Now as the three points approach coincidence the plane approaches the osculating plane of the curve and the circles common to the spheres approach circles of the annular surface.

It will be seen later that the order of the curve is $2n$ where n is the order of the developable. This curve is analogous to the cuspidal edge of the developable. The tangent lines of a plane curve are transformed into the circle generators of an annular surface. All these circles have the representative point pair of the plane of the curve in common. These last surfaces play the same rôle in this geometry that cones do in the ordinary line geometry. In particular if the plane curve is a conic the circled surface, as will appear later, will be a binodal cyclide which plays an important part in what follows.

It is at once seen that a cone is transformed into a curve traced on the sphere into which the vertex of the cone is transformed. The elements of the cone are transformed into circles which envelope the curve, and the tangent planes of the cone are transformed into the points of the curve.

This is the transformation which Laguerre and Darboux used to obtain the cyclides from the quadric surfaces. Darboux* defines a point transformation by a combination of polar reciprocation with respect to S and the trans-

* Loc. cit., p. 126.

formation described above. He derives the equations of this transformation and by means of them obtains the properties of the transformed surface. The properties of the surface obtained by applying the transformation under discussion can be obtained, however, without the use of equations.

2. Transformation of surfaces. A surface A of class n is transformed into a surface which has the circle at infinity as n-fold line. For, consider a pencil of n minimum planes, by which we mean a pencil whose axis is a line in the plane at infinity which touches the circle at infinity; n planes of this pencil will touch A, and (since the representative point pair of a minimum plane consists of the point where the plane touches the circle at infinity and a point on the line joining O to this point) *each point on the circle at infinity will be an n-fold point on the transformed surface.*

We can now obtain the order of the transformed surface. The surface A is of class n and therefore n planes of any pencil will touch A. Each of these tangent planes is tranformed into two points of the transformed surface. Moreover the axis of the pencil is transformed into a circle which contains the representative point pair of each plane of the pencil and in particular of each of the tangent planes. Hence this circle intersects the transformed surface in $2n$ finite points, and, since any circle cuts the circle at infinity in two points, we have still $2n$ points of intersection of the circle and the surface to reckon. The total number of points of intersection is therefore $4n$, and we have the theorem:

The transformed surface is of order $2n$ and contains the circle at infinity as an n-fold line.

We saw that the plane at infinity is transformed into O together with any point in the plane at infinity (that is, into the plane at infinity itself). Therefore if a surface has the plane at infinity for tangent plane the transformed surface will contain the plane at infinity as a part, i. e., the surface will be factorable, the order of the residual surface will be reduced by one, and the number of times which the surface contains the circle at infinity will also be reduced by one. Hence *if the surface A is tangent r times to the plane at infinity the transformed surface will be of order $2n - r$ and will contain the circle at infinity as an $(n - r)$-fold line.*

If the axis of the pencil of minimum planes considered above touches A, two planes of the pencil will coincide and therefore it is easily seen that two sheets of the transformed surface will be tangent at this point. The number

of such points will be equal to the number of common tangents that can be drawn to the circle at infinity and the section of A in the plane at infinity. Loria calls such points *cuspidal points*.

Surfaces obtained by transforming scrolls.

The straight lines on the ruled surface become circles on the transformed surface. The centers of these circles lie on the given ruled surface at the points of intersection of the generators and lines drawn from O perpendicular to the generators. The order of the curve of centers will be equal to the number of generators of the ruled surface whose normals drawn from O lie in a plane.

All lines whose normals drawn from O lie in a plane P can be arranged in plane pencils whose vertices lie in P and such that the line drawn from O to a vertex is normal to the plane of the corresponding pencil. All the lines of the pencil that pass through a fixed point not in P generate a cone whose section in P is a circle having O and the foot of the perpendicular dropped from the point to P as extremities of a diameter. The generators of a ruled surface establish a $(1, 1)$ correspondence between the points of any two general plane sections c_1 and c_2. If from each point of c_1 we draw the cone of lines whose normals from O lie in P, c_2 will intersect it in $2n$ points and therefore a $(1, 2n)$ correspondence is established between the points of c_1 and c_2, and vice versa. Now let A_0, B_0 be the points where a generator cuts c_1 and c_2 respectively. To A_0 corresponds $2n$ points of c_2 as has just been shown; but A_0 corresponds to B_0 (by correspondence set up by generators of the surface); hence to B_0 correspond $2n$ points of c_2, from which we see that to each point of c_2 considered as a point on a generator correspond $2n$ points considered as points on the cones, and vice versa. Therefore a $(2n, 2n)$ correspondence is established between the points of c_2, and the coincidences in general give the number of generators of the ruled surface which are also elements of the cones. But at each of the n intersections of c_1 and c_2 there are two coincidences which do not give elements of the cones, for such a point is the vertex of a cone and therefore two elements of the cone lie in the plane of c_2; these two elements correspond to the generator through the point but they do not coincide. In a $(2n, 2n)$ correspondence there are in all $4n$ coincidences but as $2n$ of these do not give elements of the cones we have:

The locus of the centers of the circles of a circled surface obtained by

transforming a ruled surface of order n is a curve of order $2n$. It can easily be shown that the planes of the circles envelope a cone of order n. The locus of the centers of the circles into which the elements of a cone are transformed is a curve traced on a sphere. It has a multiple point at the vertex of the cone.

It is known from the theory of ruled surfaces that each generator of a ruled surface cuts $n-2$ other generators; that is, each generator cuts the nodal curve in $n-2$ points. Therefore in the transformed surface each circle generator cuts $n-2$ other generators, or it cuts the nodal curve in $2(n-2)$ points. Multiple generators on the ruled surface are transformed into multiple generators of the circled surface. A multiple rectilinear directrix of order r is transformed into a circular directrix of order $n-r$, for any plane which contains the directrix will cut from the ruled surface $n-r$ generators, and therefore through each point pair of the circle into which the directrix is transformed will pass $n-r$ generators of the circled surface. Each point of the rectilinear directrix of the ruled surface is transformed into a sphere which contains the circle into which the directrix is transformed and also the circles into which the generators of the ruled surface which pass through the point are transformed. This is the complete intersection of the sphere with the circled surface. Hence, *a sphere which passes through a directrix of order* $n-r$ *on the circled surface cuts from it* r *circular generators.*

To a multiple rectilinear directrix of the ruled surface corresponds a second nodal curve on the circled surface, viz.: the curve into which the developable containing two generators passing through the same point of the directrix is transformed. To a nodal curve on a ruled surface which is not a straight line corresponds a nodal curve on the circled surface obtained by transforming the developable formed by planes containing two generators through the same point of the nodal curve. A torsal generator of the ruled surface is transformed into a torsal generator of the circled surface.

3. Congruences of circles. To every congruence of straight lines corresponds by the transformation a congruence of circles orthogonal to S, and vice versa. We will define the order of the congruence as the number of circles of the congruence which pass through an arbitrary point pair, and the class of the congruence as the number of circles which lie on an arbitrary sphere orthogonal to S. Then it follows at once from the transformation that the order and class of a congruence of circles orthogonal to a given sphere are

equal respectively to the class and order of the congruence of straight lines formed by the axes of the circles.

The focal surface. The focal points of any curve of a congruence are defined as the points of the curve in which it is cut by the curves of the congruence which are infinitely close and which cut it. The focal surface of the congruence is defined as the locus of these focal points. In a congruence of circles orthogonal to S, intersecting circles will result from transforming lines of the line congruence which intersect, and since each line of the congruence is intersected by two lines of the congruence infinitely near, and these are transformed into intersecting circles, it follows that each circle of the congruence will be cut in two points by each of two circles infinitely near. Hence, *on each circle there are four focal points, two of which are the inverses of the other two with respect to* S. These points are obtained by transforming the planes which contain the given line and one of the lines infinitely close; but such planes are tangent to the focal surface of the line congruence. Hence *the focal surface of the circle congruence is obtained by transforming the focal surface of the line congruence.*

If the line congruence consists of all lines which cut two curves or which cut a given curve in two points, the focal surface degenerates into the curves. The focal surface of the corresponding circle congruence becomes the annular surfaces into which the curves are transformed. In particular, if the line congruence is linear the focal surface consists of two non-intersecting straight lines and the focal surface of the corresponding congruence of circles becomes the pencils of spheres into which the points of the lines are transformed, i. e., the circles enveloped by these spheres; therefore we may define a linear congruence of circles as all the circles which cut two fixed non-intersecting circles.

4. The linear complex. A complex of lines is transformed into a complex of circles and in particular a linear complex of lines is transformed into a linear complex of circles. By the transformation the configurations of the linear circle complex are connected with those of the linear line complex as follows:

Line complex	*Circle complex*
Lines of the complex lying in the same plane pass through the same point called the pole of the plane.	Circles of the complex which pass through the same point pair lie on the same sphere called the polar sphere of the point pair.

Lines which pass through the same point lie in the same plane called the polar plane of the point.	Circles which lie on the same sphere pass through the same point pair called the polar point pair of the sphere.
If a plane revolves about a line its pole describes a second line called the polar conjugate.	If a point pair describe a circle the polar spheres will envelope a second circle called the polar conjugate.
Every line of the complex which cuts a given line will also cut its conjugate polar.	Every circle of the complex which cuts a given circle will also cut its conjugate polar.

The line joining a point pair always passes through O, the center of S. We will call the anharmonic ratio of four point pairs, on the same circle, the anharmonic ratio of the four lines joining them to O. Then from the (1, 1) correspondence between the point pairs and their polar spheres we see that *the anharmonic ratio of four point pairs on the same circle is the same as the anharmonic ratio of their polar spheres.*

In the remainder of the paper considerable knowledge of the configurations of lines of both the linear and quadratic line complex is presupposed. By an application of the above transformation to these configurations we obtain the configurations of the linear and quadratic circle complex. The results of this transformation will be given, but space forbids giving more than a hint of the demonstration.

The form of the linear complex. The linear line complex remains invariant with respect to rotation about or translation along its axis. Through O draw a plane π normal to the axis of the line complex. By the transformation the axis will be transformed into a circle C lying in π and such that all the circles which cut it will lie in planes perpendicular to π (since by this transformation the angle between two circles is the supplement of the angle between the corresponding lines). These planes form a pencil whose axis l passes through O and is parallel to the axis of the line complex. Planes which contain lines making the same angle with the axis of the line complex envelope a circular cylinder with the axis of the complex for axis. The lines in any one of these planes are transformed into circles lying in a plane which makes the same angle with π that the lines made

with the axis of the complex; these planes then envelope a circular cone. The circles which lie in the same enveloping plane of this cone intersect in a point pair in π and the locus of this point pair is a bicircular quartic. This quartic is the envelope of circles, in π, whose centers lie on the circle which π cuts from the corresponding cylinder and which are orthogonal to S. Therefore *a linear circle complex consists of those circles which lie in planes enveloping a series of coaxial circular cones and whose points of intersection lie on a bicircular quartic.*

Analogous to Sylvester's method of constructing the linear line complex, the linear circle complex consists of all circles which intersect two corresponding circles of two projective pencils which have a common self-corresponding circle.

Linear series. Lines common to three linear line complexes form one generation of the hyperboloid; therefore circles common to three linear circle complexes will be the surface into which the hyperboloid is transformed. This is the generation of the cyclide given by Laguerre.*

Since four generators of each system of the hyperboloid are tangent to S, and to each of these corresponds two minimum lines, the 16 minimum lines of the cyclide are degenerate circles. The planes of these degenerate circles pass through O and their centers (points of intersection of the minimum lines) lie on the sphero-quartic in which S intersects the hyperboloid. Since each generator of one system of the cyclide intersects all the circles of the other system, each minimum line will intersect five others. The cyclide can be generated in five ways,† as above; therefore *the points of intersection of the minimum lines can be arranged in five groups of eight which lie on a sphero-quartic and such that the planes containing the pairs of lines of any group pass through a common point.* The locus of centers of the circles of one generation of the cyclide is a quartic curve which intersects the sphere S in the four points of intersection of the four pairs of minimum lines belonging to the generation. It is known that there are ten systems of circles on a cyclide, and Laguerre has shown‡ that the other systems are obtained by transforming the tangent cones of the quadric drawn from points of the double curve of the developable circumscribing S and the quadric.

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* *Œuvres*, p. 53.

† See Laguerre, *Œuvres*, p. 53.

‡ *Œuvres*, vol. II, p. 57.

Three non-intersecting lines determine a hyperboloid and therefore three non-intersecting circles orthogonal to the same sphere determine a cyclide uniquely. Any cyclide except Dupin's can appear as the surface common to three linear complexes if the quadric has a particular relation to S. The surfaces common to three linear complexes must have two distinct generations of circles, however; e. g., the binodal cyclide can be one if it is obtained by transforming a quadric which is tangent to S at two points, but not if it is considered as the surface into which the tangent lines to a conic are transformed. The cyclide may degenerate into two pencils of circles which have a circle in common.

5. The quadratic complex. A quadratic complex of lines is transformed into a quadratic complex of circles. The transformation is as follows:

Line complex.	*Circle complex.*
Lines of the complex which pass through a point generate a cone called the complex cone.	Circles of the complex which lie on a sphere envelope a sphero-quartic called the complex quartic.
Lines which lie in the same plane envelope a conic called the complex conic.	Circles of the complex which pass through a point pair generate a binodal cyclide called the complex cyclide.
The surface of singularities is the locus of those points whose complex cone breaks up into two planes, or the envelope of those planes whose complex conic breaks up into two pencils.	The surface of singularities is the envelope of those spheres whose complex quartic breaks up into two pencils, or the locus of those point pairs whose complex cyclide breaks up into two spheres.

The surface of singularities of the quadratic line complex is the Kummer surface; the surface of singularities of the quadratic circle complex is obtained by transforming this surface. This transformed surface is of order 8; it contains the circle at infinity as a four-fold line, and has eight cuspidal points on the circle at infinity. The curve of intersection of S with the Kummer surface is a focal line on the surface; the remainder of the focal curve is the double line of the developable circumscribed to this curve and the circle at infinity. There are 16 spheres which have singular contact with the surface

along a sphero-quartic; these 16 spheres are grouped in sixes so that the centers of the six spheres of a group lie on a conic; the six spheres of a group have two points in common; six of these planes of centers pass through the center of each sphere. The surface has 16 pairs of conical points derived from the 16 singular tangent planes of the Kummer surface.

The surfaces of singularities of the special forms of the quadratic complex are obtained by transforming the corresponding Kummer surface.

APPENDIX.

A. Transformation of a minimum plane.

If (x, y, z, t) are the homogeneous coordinates of a point, the equation of a minimum plane is

$$Ax + By + Cz + Dt = 0,$$

where the coefficients satisfy the relation

$$A^2 + B^2 + C^2 = 0.$$

The pencil of spheres defined by this plane and S, whose equation is

$$S = x^2 + y^2 + z^2 - R^2t^2 = 0,$$

is

$$(1) \qquad l(x^2 + y^2 + z^2 - R^2t^2) + 2kt(Ax + By + Cz + Dt) = 0,$$

or, rewriting,

$$(2) \qquad (lx + kAt)^2 + (ly + kBt)^2 + (lz + kCt)^2 = l^2R^2t^2 - 2klDt^2.$$

The point spheres are given by the values of l and k which satisfy

$$(3) \qquad l^2R^2t^2 - 2klDt^2 = 0.$$

The roots of this equation are $l = 0$ and $k/l = R^2/2D$. From equation (1) we see that the point sphere corresponding to $l = 0$ is

$$t(Ax + By + Cz + Dt) = 0,$$

which degenerates into $t = 0$ and $Ax + By + Cz + Dt = 0$. Thus it is seen that in a pencil of spheres defined by a minimum plane and S, one

of the point spheres degrades into the minimum plane itself together with the plane at infinity. The centers of the spheres of the pencil are the points $x: y: z: t = kA: kB: kC: l$ (see equation 2), and when $l = 0$ the point becomes $x: y: z = A: B: C, t = 0$; i. e., the center of the degraded point sphere is the point where the minimum plane touches the circle at infinity. In this sense it is seen that a minimum plane taken with the plane at infinity is a point sphere and that its center is on the circle at infinity. The center of the other point sphere of the pencil is

$$\left(\frac{AR^2}{2D}, \frac{BR^2}{2D}, \frac{CR^2}{2D}\right)$$

which is seen to be on the line passing through O and having direction cosines proportional to A, B, C. This is the line joining O to the point where the minimum plane touches the circle at infinity.

B. Transformation of the plane at infinity.

The equation of the plane at infinity is

$$Ax + By + Cz + Dt = 0, \quad \text{where} \quad A = B = C = 0.$$

From equation (3) above the point spheres of the pencil determined by this plane and S are given by the solutions of

$$R^2t^2l^2 - 2Dtkl = 0,$$

which are $k/l = R^2/2D$ and $l = 0$.

The center of the point sphere corresponding to $R^2/2D$ is $(0, 0, 0)$ [see equation (2) of appendix A]. The point sphere corresponding to $l = 0$ is the plane at infinity counted twice. Its center is $(kA/l, kB/l, kC/l)$, which is illusory since $A = B = C = l = 0$. This means that the center of the point sphere which degrades into the plane at infinity counted twice is any point whatever in the plane at infinity.

C. Transformation of a point in the plane at infinity.

The sphere S' into which any point $(\xi: \eta: \zeta: 1)$ is transformed is

$$S' \equiv (x - \xi t)^2 + (y - \eta t)^2 + (z - \zeta t)^2 - r^2t^2 = 0,$$

where $r^2 = \xi^2 + \eta^2 + \zeta^2 - R^2$ (since S and S' are orthogonal). Expanding and making use of the above relation, we have

$$x^2 + y^2 + z^2 - 2t(\xi x + \eta y + \zeta z) + R^2 t^2 = 0,$$

which we may write

$$\frac{1}{\sqrt{\xi^2 + \eta^2 + \zeta^2}} [x^2 + y^2 + z^2 - 2t\,(\xi x + \eta y + \zeta z) + R^2 t^2] = 0.$$

Now as the point $(\xi : \eta : \zeta : 1)$ recedes to infinity along a line passing through O,

$$\frac{\xi}{\sqrt{\xi^2 + \eta^2 + \zeta^2}}, \quad \frac{\eta}{\sqrt{\xi^2 + \eta^2 + \zeta^2}}, \quad \frac{\zeta}{\sqrt{\xi^2 + \eta^2 + \zeta^2}}$$

remain finite and the last equation reduces to

$$\frac{\xi x + \eta y + \zeta z}{\sqrt{\xi^2 + \eta^2 + \zeta^2}} = 0, \quad t = 0.$$

The sphere therefore reduces to a plane passing through O perpendicular to the line whose direction cosines are proportional to ξ, η, ζ, together with the plane at infinity.

D. Transformation of a line in the plane at infinity.

For convenience take a line perpendicular to the xy-plane, whose equations are $x = \xi$, $y = \eta$. The circle into which it is transformed is

$$(x - \xi t)^2 + (y - \eta t)^2 - r^2 t^2 = 0, \; z = 0,$$

where $R^2 = \xi^2 + \eta^2 - r^2$, since the circle is orthogonal to S. Expanding and making use of this relation, we have

$$x^2 + y^2 - 2t\,(\xi x + \eta y) + R^2 t^2 = 0, \quad z = 0, \tag{4}$$

$$\frac{1}{\sqrt{\xi^2 + \eta^2}} [x^2 + y^2 - 2t(\xi x + \eta y) + R^2 t^2] = 0, \quad z = 0. \tag{5}$$

Now as the line recedes to infinity in a definite direction, $\frac{\xi}{\sqrt{\xi^2+\eta^2}}$ and $\frac{\eta}{\sqrt{\xi^2+\eta^2}}$ remain constant and the last equation reduces to

$$\frac{\xi x+\eta y}{\sqrt{\xi^2+\eta^2}}=0, \quad t=0; \quad z=0,$$

which shows that the circle reduces to the lines where the plane $z=0$ cuts the planes $\xi x+\eta y=0$, and $t=0$; i. e., it is transformed into a line through O together with a line in the plane at infinity.

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ON FUNCTIONAL DETERMINANTS

By Paul Saurel

The following theorem is well known:* If n functions $u_1, u_2, \cdots, u_n$ of the n variables $x_1, x_2, \cdots, x_n$ are connected by a relation

$$f(u_1, u_2, \cdots, u_n) = 0,$$

the functional determinant

$$\frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)}$$

vanishes; and, conversely, if the functional determinant vanishes, there exists a relation connecting the n functions.

In the present note we shall give a demonstration of the following more general but less known theorem:† If r functions $u_1, u_2, \cdots, u_r$ of the n variables $x_1, x_2, \cdots, x_n$, r being less than or equal to n, are connected by a relation

$$f(u_1, u_2, \cdots, u_r) = 0, \tag{1}$$

every determinant of order r formed from the matrix

$$\begin{matrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial u_r}{\partial x_1} & \frac{\partial u_r}{\partial x_2} & \cdots & \frac{\partial u_r}{\partial x_n} \end{matrix} \tag{2}$$

vanishes; and, conversely, if every determinant of order r formed from the

* C. G. J. Jacobi, *Journal für die reine und angewandte Mathematik*, vol. 22, p. 328; 1841. Ostwald's *Klassiker der exacten Wissenschaften*, No. 78, Ueber die Functionaldeterminanten, p. 16.

† Loc. cit., p. 347 or p. 42.

above matrix vanishes while in each of the r matrices which can be derived from the given matrix by suppressing one row there is at least one determinant of order $r-1$ which does not vanish, there exists a relation connecting the r functions and involving all of them.

The demonstration of the first part of the theorem is very simple. If equation 1 holds, then the n equations

$$\sum_{i=1}^{r} \frac{\partial f}{\partial u_i} \frac{\partial u_i}{\partial x_j} = 0 \qquad (j = 1, 2, \cdots, n) \qquad (3)$$

obtained by differentiating equation 1 will also hold. Now, since each of the r functions is supposed to appear in the given relation 1, it follows that no one of the expressions $\frac{\partial f}{\partial u_i}$ is identically equal to zero; it then follows at once from equations 3 that every determinant of order r formed from the matrix 2 must be equal to zero.

Let us now suppose that every determinant of order r formed from the matrix 2 is equal to zero and that in each of the matrices obtained by suppressing one row of the given matrix there is at least one determinant of order $r-1$ which is not equal to zero. Since in the matrix which is formed by suppressing the last row in (2) there is at least one determinant of order $r-1$ which is not equal to zero, we may assume, without loss of generality, that

$$\frac{\partial(u_1, u_2, \cdots, u_{r-1})}{\partial(x_1, x_2, \cdots, x_{r-1})} \neq 0. \qquad (4)$$

From this it follows that at least one of the expressions $\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \cdots, \frac{\partial u_1}{\partial x_{r-1}}$ is not equal to zero; we may therefore assume, again without loss of generality, that

$$\frac{\partial u_1}{\partial x_1} \neq 0. \qquad (5)$$

Inequality 5 shows that u_1 depends upon x_1; it is therefore possible to express x_1 as a function of $u_1, x_2, x_3, \cdots, x_n$. Using this value of x_1 we can then express $u_2, u_3, \cdots, u_r$ as functions of $u_1, x_2, x_3, \cdots, x_n$. If we denote the resulting expressions by $u_2', u_3', \cdots, u_r'$ we get without difficulty

$$\begin{aligned} \frac{\partial u_i}{\partial x_1} &= \frac{\partial u_i'}{\partial u_1}\frac{\partial u_1}{\partial x_1} && (i = 2, 3, \cdots, r), \\ \frac{\partial u_i}{\partial x_j} &= \frac{\partial u_i'}{\partial u_1}\frac{\partial u_1}{\partial x_j} + \frac{\partial u_i'}{\partial x_j} && (j = 2, 3, \cdots, n). \end{aligned} \tag{6}$$

If we substitute these values in (4) and change the form of the resulting determinant by subtracting from the elements of the i^{th} row those of the first row multiplied by $\frac{\partial u_i'}{\partial u_1}$, we find that

$$\frac{\partial u_1}{\partial x_1} \cdot \frac{\partial (u_2', u_3', \cdots, u_{r-1}')}{\partial (x_2, x_3, \cdots, x_{r-1})} \neq 0. \tag{7}$$

From (5) and (7) it follows that

$$\frac{\partial (u_2', u_3', \cdots, u_{r-1}')}{\partial (x_2, x_3, \cdots, x_{r-1})} \neq 0. \tag{8}$$

In like manner formulas 5 and 6 will enable us to replace the equations

$$\frac{\partial (u_1, u_2, \cdots, u_{r-1}, u_r)}{\partial (x_1, x_2, \cdots, x_{r-1}, x_{r+s})} = 0 \qquad (s = 0, 1, 2, \cdots, n - r), \tag{9}$$

by the equivalent equations

$$\frac{\partial (u_2', u_3', \cdots, u_{r-1}', u_r')}{\partial (x_2, x_3, \cdots, x_{r-1}, x_{r+s})} = 0 \qquad (s = 0, 1, 2, \cdots, n - r). \tag{10}$$

If we remember that when equations 9 hold every determinant of order r formed from the matrix 2 vanishes, it follows from (10) that every determinant of order $r - 1$ formed from the matrix

$$\begin{matrix} \frac{\partial u_2'}{\partial x_2} & \frac{\partial u_2'}{\partial x_3} & \cdots & \frac{\partial u_2'}{\partial x_n} \\ \frac{\partial u_3'}{\partial x_2} & \frac{\partial u_3'}{\partial x_3} & \cdots & \frac{\partial u_3'}{\partial x_n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial u_r'}{\partial x_2} & \frac{\partial u_r'}{\partial x_3} & \cdots & \frac{\partial u_r'}{\partial x_n} \end{matrix} \tag{11}$$

vanishes.

Starting now with the matrix 11 and the inequality 8, and repeating the reasoning which we applied to the matrix 2 and the inequality 4, we can show that

$$\frac{\partial(u_3'', u_4'', \cdots, u_{r-1}'')}{\partial(x_3, x_4, \cdots, x_{r-1})} \neq 0, \tag{12}$$

and that every determinant of order $r-2$ formed from the matrix

$$\begin{matrix} \frac{\partial u_3''}{\partial x_3} & \frac{\partial u_3''}{\partial x_4} & \cdots & \frac{\partial u_3''}{\partial x_n} \\ \frac{\partial u_4''}{\partial x_3} & \frac{\partial u_4''}{\partial x_4} & \cdots & \frac{\partial u_4''}{\partial x_n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial u_r''}{\partial x_3} & \frac{\partial u_r''}{\partial x_4} & \cdots & \frac{\partial u_r''}{\partial x_n} \end{matrix} \tag{13}$$

vanishes. Here $u_3'', u_4'', \cdots, u_r''$ denote the functions of $u_1, u_2', x_3, \cdots, x_n$ which are obtained from $u_3', u_4', \cdots, u_r'$ by replacing x_2 by its value in terms of $u_1, u_2', x_3, \cdots, x_n$.

By repeating this process we shall finally arrive at the result that every term in the matrix

$$\frac{\partial u_r^{(r-1)}}{\partial x_r} \quad \frac{\partial u_r^{(r-1)}}{\partial x_{r+1}} \cdots \frac{\partial u_r^{(r-1)}}{\partial x_n} \tag{14}$$

must be equal to zero. As $u_r^{(r-1)}$ denotes the expression which is obtained when u_r is expressed in terms of $u_1, u_2', u_3'', \cdots, u_{r-1}^{(r-2)}, x_r, x_{r+1}, \cdots, x_n$, it follows that $u_r^{(r-1)}$ is a function of $u_1, u_2', u_3'', \cdots, u_{r-1}^{(r-2)}$ alone. Dropping the primes we may therefore say that there exists a relation connecting the r functions $u_1, u_2, \cdots, u_r$. Moreover this relation must involve each of the r functions. For, if one of them were missing from it, every determinant of order $r-1$ formed from the matrix corresponding to the remaining $r-1$ functions would vanish; but this contradicts the hypothesis. The proposition is thus established.

NEW YORK, DECEMBER, 1905.

INVOLUTORY TRANSFORMATIONS IN THE PROJECTIVE GROUP AND IN ITS SUBGROUPS*

By Edwin Bidwell Wilson

The importance of simple transpositions in the general theory of substitutions would naturally lead to the investigation of the corresponding geometric problem of the relation of involutory transformations† to the groups in which they occur; and in fact a considerable number of researches upon this subject have been published. Several mathematicians, obviously under the influence of Cremona, have taken up the involutory transformations of the Cremona type in all generality, and Kantor has even extended the field of operations to Cremona transformations of period higher than two.‡ On the other hand Study and Wiener, working in the early nineties, chose a more special field and brought the theory of the group of motions and transformations of symmetry in its relation to the allied involutory transformations to a high state of detailed development.§

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* Presented to the American Mathematical Society, September 4, 1906, as the introduction to a paper on "Oblique reflections and unimodular strains."

† An involutory transformation may be defined as one of period 2; and a transformation is said to be of period p when p is the smallest number such that a p-fold repetition of the transformation gives the identical transformation, that is, $T^p = 1$ or $T^{p-1} = T$.

‡ After the initial work of Cremona and Jonquières, there followed a large number of papers on periodic and in particular involutory bi-rational transformation. It will be sufficient to cite memoirs by Bertini in the *Rendiconti of the Royal Institute of Lombardy* for the years 1883 and 1889, where ample references to preceding work are given: and Kantor's *Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene*, Berlin, 1895.

§ E. Study, "Von den Bewegungen und Umlegungen," *Mathematische Annalen*, vol. 39, pp. 441-566 (1891). H. Wiener, a series of papers in the *Leipziger Berichte*, vols. 42, 43, 45 (1890–93).

This work was taken up later by Gale and Wood, the latter of whom extended it to cover non-Euclidean motions.* Smith then attacked the classical problem of the transformation of a quadratic form into itself, and solved it by means of central reflections.† If the quadratic surface obtained by setting the form equal to zero be considered as the absolute in a non-Euclidean geometry, it appears that the work done by Smith is, in its geometrical aspect, still essentially confined to the group of (non-Euclidean) motions and transformations of symmetry (in any number of dimensions).

In regard to the study of the general group of collineations from the point of view of its involutory transformations very little of a detailed nature has been accomplished. Some years ago I published a short note concerning the relation of the general collineation in space of three dimensions to skew reflections;‡ but as Smith subsequently showed, this question is much better treated by introducing line coordinates and regarding the problem as one in non-Euclidean geometry of five dimensions.§ After introducing the conception of area (or volume) relative to a line (or plane), I treated in some detail those collineations of the plane which leave invariant the area relative to some line.‖ Recently Haskell has obtained an existence theorem for the decomposition into central reflections of any n-dimensional collineation which has a complete set of fixed points; but he leaves the geometry of the problem quite untouched.¶

The object of the present communication is to set forth systematically certain elementary properties of involutory projective transformations which are at the basis of the investigations mentioned above and which shall serve as an introduction to further work. It may be stated that there is an algebraic

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* A. S. Gale, "Wiener's theory of displacements, etc.," ANNALS OF MATHEMATICS, ser. 2, vol. 2, pp. 1–7 (1900). R. G. Wood, "The collineations of space which transform a non-degenerate quadric surface into itself," *ibid.*, pp. 161–171 (1901).

† P. F. Smith, "On the linear transformations of a quadric into itself," *Trans. Amer. Math. Soc.*, vol. 6, pp. 1–16 (1905).

‡ E. B. Wilson, *Trans. Amer. Math. Soc.*, vol. 1, pp. 193–196 (1900).

§ P. F. Smith, *loc. cit.*, p. 15.

‖ E. B. Wilson, "Ueber eine vom Begriffe der Länge unabhangige Definition des Volumens," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 12, pp. 555–561 (1903); and "A generalised conception of area; applications etc.," ANNALS OF MATHEMATICS, ser. 2, vol. 5, pp. 29–45 (1903).

¶ M. W. Haskell, "The resolution of any collineation into perspective reflections," *Trans. Amer. Math. Soc.*, vol. 7, pp. 361–369 (1906).

as well as geometric interest which the present problem may serve. The transformation

$$\begin{aligned}
\rho x_1' &= a_{11}x_1 + \cdots + a_{1\,n+1}x_{n+1} \\
\rho x_2' &= a_{21}x_1 + \cdots + a_{2\,n+1}x_{n+1} \\
&\cdots\cdots\cdots\cdots \\
\rho x_n' &= a_{n1}x_1 + \cdots + a_{n\,n+1}x_{n+1} \\
\rho x_{n+1}' &= a_{n+1\,1}x_1 + \cdots + a_{n+1\,n+1}x_{n+1}
\end{aligned}$$

may be regarded in a double aspect. If the x's are interpreted as homogeneous point coordinates in space of n dimensions, the equations are those of the general projective transformation in this space; but if the x's are interpreted as Cartesian coordinates of the points in space of $n+1$ dimensions (and a fixed value, say unity, is given to ρ) the equations become those of the general homogeneous strain in $n+1$ dimensions. This strain may be regarded as a multiple quantity in $(n+1)^2$ units. The product of two strains will be given by the properly chosen product of the corresponding multiple quantities.* As the identical transformation is represented by the idemfactor I, the involutory transformations must be the geometric counterparts of the square roots of I (or, in the projective group, of kI).

Thus, *the involutory transformations of the linear group are interesting not only because they are the projective generalisations of the various kinds of ordinary physical reflections and because they are the natural generalisations of simple transpositions, but also for the reason that they correspond to the simplest roots of the simplest multiple quantity—the square root of the idemfactor, which is but the generalisation of unity.*†

1. The projective involutory transformations. It is well known that the various physical reflections and their projective generalisations are involutory collineations. It seems to be taken for granted that these are the only collineations which are involutory—which indeed is true. The proof of this fact has been given by Bertini;‡ but as no subsequent writers refer to his

* See Gibbs, "On multiple algebra," *Proceedings of the American Association for the Advancement of Science*, vol. 35, 1886.

† The detailed development of this aspect of the relation of involutory transformations to the general projective group is reserved for a later paper on "Oblique reflections and unimodular strains."

‡ Bertini, in the *Rendiconti of the Royal Institute of Lombardy*, vol. 19, pp. 176–183 (1886) under the title "Le omografie involutorie in uno spazio lineare a qualsivoglia numero di dimensioni."

paper, a somewhat different demonstration of the theorem may be given here for the sake of completeness and in order that we may have more definitely in mind the fundamental geometric entity with which we deal. The proof is by induction, and depends on the theorem that any collineation in n-dimensional space has at least one fixed linear space of $n-1$ dimensions.* (If it were assumed that an involutory transformation had a complete set of fixed points, that is, in space of n dimensions $n+1$ linearly independent fixed points, the proof would be immediate and obvious. It is, however, our purpose throughout this work to avoid the vague assumption that we are dealing with the "general" case—an assumption which is the bane of a great deal of geometric work).†

Consider first the case in one dimension. By virtue of the fundamental theorem which has been assumed, any linear transformation of a line must have one fixed point. If this be taken as the point "infinity" of a projective number system upon the line, the transformation becomes

$$T: \qquad x' = ax + b$$

and
$$T^2: \qquad x'' = ax' + b = a^2x + b(a+1) = x.$$

Hence
$$a^2 = 1 \quad \text{and} \quad b(a+1) = 0.$$

If now $a = +1$, it follows that $b = 0$, and hence T reduces to $x' = x$ which is the identical transformation, obviously though trivially involutory. If on the other hand $a = -1$, a change of the origin to the point $b/2$ brings the equation of transformation to the form

$$T: \qquad x' = -x \qquad \text{or} \qquad \rho x_1' = -x_1, \qquad \rho x_2' = +x_2.‡$$

Thus the transformation has two fixed points, the origin and infinity, and is merely the well known involution on a line. *A similar result obviously*

* See Muth, *Theorie der Elementartheiler*, p. 200. Or better, merely note that if the x''s be set equal to the x's in the equations of the transformation given above and if the condition be expressed that the resulting $n+1$ equations have a solution for the x's, a value of ρ may be obtained which will satisfy the equations and thus a fixed point of the transformation is found. By duality, there must be a fixed linear space of $n-1$ dimensions.

† See "A problem in analytic geometry, with a moral," by M. Bôcher, ANNALS OF MATHEMATICS, ser. 2, vol. 7, pp. 44-49 (1905).

‡ A proof quite similar to this may be applied to any periodic transformation of order p. In fact the formulas for T^p are: $x^{(p)} = a^p x + b(a^{p-1} + a^{p-2} + \cdots + a + 1) = x$. Hence a is some p^{th} root of unity. If $b = 0$, T is already of the type desired; if not, a transformation of the origin to the point $b/(1-e)$, where e is the p^{th} root of unity, brings T to the type $\rho x_1' = ex_1$, $\rho x_2' = x_2$. Hence there are two fixed points.

holds in the case of a pencil of lines or of planes, or any set which depends on one linear parameter and which undergoes an involutory transformation.

In the case of the plane one line must be fixed. It may be fixed point for point or only as a whole—in which case the transformation upon it must be involutory. In either case there are at least two distinct fixed points on the line. Consider the pencils which have their vertices at these points. Each of them is transformed either identically or in an involutory fashion. Hence, as has been seen above, there must be not less than two fixed rays in each pencil, one in addition to the assumed fixed line. These additional rays intersect in a point which is fixed. Hence in the plane the involutory transformation has a complete set of fixed points. Passing on to space we note that there must be at least one fixed plane by virtue of the assumed fundamental theorem. Whether this plane is transformed identically or in involutory fashion, it contains a complete system of three fixed points joined by three fixed lines. If the same reasoning be applied to the pencils of planes which have these lines as axes as was applied to the pencils in the case of the plane, it becomes evident that there is a fourth fixed point not in the plane which was assumed as fixed. Hence the complete set of points for space; and so on to any number of dimensions.

The equations of any collineation in n dimensions referred to a system of fixed points are

$$\begin{aligned} T:\quad & \rho x_1' = a_1 x_1 \\ & \rho x_2' = a_2 x_2 \\ & \cdots\cdots \\ & \cdots\cdots \\ & \rho x_n' = a_n x_n \\ & \rho x_{n+1}' = a_{n+1} x_{n+1} \end{aligned} \qquad \begin{aligned} T^2:\quad & \rho x_1'' = a_1^2 x_1 \\ & \rho x_2'' = a_2^2 x_2 \\ & \cdots\cdots \\ & \cdots\cdots \\ & \rho x_n'' = a_n^2 x_n \\ & \rho x_{n+1}'' = a_{n+1}^2 x_{n+1}. \end{aligned}$$

Hence if T is involutory the a's are square roots of unity and the equations may be written in the form

$$\rho x_1' = \pm x_1,\ \rho x_2' = \pm x_2,\ \cdots \rho x_n' = \pm x_n,\ \rho x_{n+1}' = \pm x_{n+1}.$$*

* The proof may be extended to transformations of period p. No step has to be altered, except that the preceding foot-note must replace the work in the text. The result is that any transformation of period p may be written in the form

$$\rho x_1' = e_1 x_1,\ \rho x_2' = e_2 x_2,\ \ldots,\ \rho x_n' = e_n x_n,\ \rho x_{n+1}' = e_{n+1} x_{n+1},$$

where $e_1, e_2, \ldots, e_{n+1}$ are any p^{th} roots of unity.

There are many different combinations of sign which are possible, but owing to the possibility of permuting the letters and of multiplying through by a factor, the number of essentially different involutory projective transformations is only $E\left(\frac{n+1}{2}\right)$, where $E(x)$ denotes the integral part of x. If the first k signs be taken negative and the remaining $n-k+1$ be positive, the transformation takes the form

$$\rho x_1' = -x_1,\ \rho x_2' = -x_2,\ \cdots,\ \rho x_k' = -x_k,$$

$$\rho x_{k+1}' = x_{k+1},\ \cdots,\ \rho x_n' = x_n,\ \rho x_{n+1}' = x_{n+1}.$$

The linear spaces defined by

$$S_{n-k}:\quad x_1 = x_2 = \cdots = x_k = 0,$$

$$S_{k-1}:\quad x_{k+1} = \cdots = x_n = x_{n+1} = 0$$

remain identically fixed, as the ratios of the coordinates of their points are not changed. The coordinates of any pair of points which interchange their positions are

$$x_1 : x_2 \quad : \quad \cdots \quad : \quad x_k : x_{k+1} : \quad \cdots \quad : x_n : x_{n+1},$$

$$-x_1 : -x_2 : \quad \cdots \quad : -x_k : x_{k+1} : \quad \cdots : \quad x_k : x_{k+1}.$$

These points are collinear with and harmonically situated with repect to the two points

$$0 : 0 : \quad \cdots \quad : 0 : x_{k+1} : \quad \cdots \quad : x_n : x_{n+1},$$

$$x_1 : x_2 : \quad \cdots \quad : x_k : \ 0 \ : \quad \cdots \quad : 0 : \ 0,$$

which lie respectively in the spaces S_{n-k} and S_{k-1}. As this is precisely the definition of a projective reflection, *it appears that the only involutory collineations are the projective reflections of the* $E\left(\frac{n+1}{2}\right)$ *different types above found.* That the projective reflection of the points of S_n in any two non-intersecting spaces S_{k-1} and S_{n-k} is involutory and definite, is obvious.

If the symbol $T(k-1, n-k)$ denote a projective reflection of S_n in the two non-intersecting spaces S_{k-1} and S_{n-k}, the number of degrees of freedom of T is the sum of the numbers of degrees of freedom of the spaces

S_{k-1} and S_{n-k} in S_n. Now the number of degrees of freedom of S_i in S_n is

$$n(i+1) - (i+1)i = (n-i)(i+1),$$

because it takes $i+1$ points each with n coordinates to determine S_i, while at the same time each of these $i+1$ points has i degrees of freedom. Hence $T(k-1, n-k)$ has

$$(n-k+1)k + k(n-k+1) = 2k(n-k+1) = 2(k-1+1)(n-k+1)$$

degrees of freedom. The following table may therefore be constructed in which the first column gives the number of dimensions, the second the type of the reflection, the third its number of degrees of freedom, the fourth the number of degrees of freedom in the general collineation, the fifth the number of involutory collineations required for the resolution of the collineation, and the sixth the remaining degrees of freedom.

$n=1$	$T(0, 0)$	2	3	2	1
$n=2$	$T(0, 1)$	4	8	3	4
$n=3$	$T(0, 2)$	6	15	4	9
	$T(1, 1)$	8	15	3	9
$n=n$	$T(0, n-1)$	$2n$	$n(n+2)$	$n+1$	n^2
	$T(k-1, n-k)$	$2k(n-k+1)$	$n(n+2)$	?	?
n odd	$T\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$	$\frac{(n+1)^2}{2}$	$n(n+2)$	3?*	$\frac{(n+1)^2}{2}+1$
n even	$T\left(\frac{n}{2}-1, \frac{n}{2}\right)$	$\frac{n(n+2)}{2}$	$n(n+2)$	3?	$\frac{n(n+2)}{2}$

So much for the general theory.

2. The discussion of the subgroups. The number of types of involutory collineations in the general projective group has been seen to be

* It seems probable that if the projective reflection which has the greatest number of degrees of freedom be chosen, three will suffice for the resolution of the general collineation. The proof has thus far been given only in the case of $n \leqq 3$; see p. 78.

$E\left(\frac{n+1}{2}\right)$. In the various subgroups the number of types may be the same or less or more. For each of these subgroups is specified by the addition of certain invariants. The problem of determining all the involutory transformations in any subgroup of the projective group is merely that of combining in all possible distinct ways the fixed spaces of the reflections with the invariants of the special group. This may be illustrated as follows.

In the group of motions in the plane the two circular points are individually fixed. Hence two cases may arise. Either the line at infinity is the fixed line of the reflection $T(0, 1)$ and the fixed point any point of the plane, or one of the circular points is the fixed point, that is, the center of the reflection, and the fixed line or axis of the reflection passes through the other. The first case gives rise to the ordinary physical reflection in a point or, in other words, to rotation about the point through the angle of 180°. The second case gives a transformation which may be written with the Cartesian equations

$$x' = iy,\ y' = -ix,\ i = \sqrt{-1}.$$

The use of this transformation for studying the Euclidean group seems slight. The product of any number of physical reflections in points produces merely a translation and therefore cannot suffice for the general discussion of the Euclidean group. It may be noted that the identical invariance of the circular points does not restrict the group to that of Euclidean motions, but allows dilatations to enter. As the reflections above mentioned cannot introduce any dilatation, it is not necessary to trouble with that additional invariant of the Euclidean group which serves to shut out the dilatations.

If the set of transformations of symmetry be adjoined to the Euclidean group, the resulting group leaves the circular points invariant merely as a pair. Hence in addition to the foregoing cases the center of the reflection may be any point on the line at infinity and the fixed line (which must cut the line at infinity in a point harmonically situated to the center with respect to the circular points) will take a direction perpendicular to the direction leading to that point which was chosen as center. The resulting involutory transformation is then merely common orthogonal or physical reflection in the fixed line. The relation of this reflection to the group in question has been taken up elsewhere.* Passing on to the more general group which leaves a specified line in-

* See Study and Wiener, *loc. cit.*

variant (the line may be taken as the line at infinity and the group then becomes the affine group) we note that there are also in this case two kinds of reflection, central and oblique reflection. The former arises when the fixed line of the group coincides with the axis of the reflection; the second when the center is situated on the fixed line. As a matter of fact the group which may be made up of such reflections leaves area with respect to the fixed line invariant.*

In a similar manner in space of three dimensions there is seen to be only one type of reflection in the Euclidean group. This is orthogonal reflection in a line or rotation about the line through 180°. Thus the Euclidean group contains fewer types of involutory transformations than the general group of all collineations. On the other hand if the group be extended to include all transformations of symmetry, there are three types of reflections—reflection in a point and orthogonal reflections whether in a line or in a plane. In this case the total number is greater than in the projective group. These groups have all been treated by other authors.† The natural generalisation of the Euclidean group to the non-Euclidean deserves a word and may be mentioned in the general case of any number of dimensions. The necessary and sufficient condition that a projective reflection shall leave a quadric surface invariant is that the fixed spaces S_{k-1} and S_{n-k} be polar with respect to the surface. Thus the number of types of non-Euclidean involutory transformations is always the same as the number in the general projective group in the same number of dimensions.‡

The next step in the development of the theory of involutory projective transformations is concerned with the unimodular strains. The restriction to unimodular strains is a necessary consequence of the invariance of volume under involutory strain. As the plane at infinity and the origin are to remain fixed, three and only three distinct types of involutory strain exist. If the projective transformation be of type $T(0, 2)$ the fixed plane may be taken at infinity and the fixed point at the origin. This gives rise to reflection through the origin. The direction of all vectors issuing from the origin is reversed. It is also possible that the fixed plane pass through the origin while the fixed point is any point at infinity. In this case the transformation becomes oblique reflection across the plane and in the direction leading to the fixed center at infinity.

* See Wilson, ANNALS OF MATHEMATICS, *loc. cit.*

† Study, Wiener, and Gale, *loc. cit.*

‡ See Smith and Wood, *loc. cit.*

If vectors be resolved into two components, one along the fixed plane of the reflection and the other along the direction leading to the center at infinity, the former components will remain unchanged while the latter are reversed in direction. If the projective reflection be of type $T(1, 1)$ one of the fixed lines may be arbitrarily chosen in the plane at infinity and the other must then pass through the origin with no other restriction save that it shall not be complanar with the first. If a plane be passed through the origin and the fixed line at infinity and if vectors be resolved parallel to this plane and to the fixed line passing through the origin, it is seen that the transformation consists in leaving the latter components unchanged and in reversing the directions of the former.*

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* The detailed treatment of the relation between these three involutory strains and the general unimodular strain is interesting from a number of points of view, geometric, algebraic, and physical. It will form the subject of another paper

ON THE CONVERGENCE AND DIFFERENTIATION OF TRIGONOMETRIC SERIES

By W. C. Brenke

In this paper are given extensions of two theorems, one due to Schlömilch,* relating to the convergence, the other due to Lerch,† relating to the differentiation of trigonometric series, the mode of proof being adapted to the case in hand from an article on "Fourier's Series" by Professor Bôcher.‡ Some examples are appended.

1. Consider the trigonometric series

$$\sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{1}$$

A sufficient condition for the convergence of series (1) is the absolute convergence of the series of its coefficients. A more general sufficient condition is given by the following theorem:

The series (1) *converges uniformly in any interval which does not include or reach up to a root of the equation*

$$\cos \frac{h}{2}(x - t) = 0,$$

where h *is a fixed integer and* t *any real constant, provided that*

$$(\alpha) \qquad \lim a_n = 0, \qquad \lim b_n = 0$$

and the series

$$(\beta) \qquad \sum (|A_n| + |B_n|)$$

converges, where

$$A_n = (a_{n-h} - a_n) \sin \tfrac{1}{2} ht + (b_{n-h} + b_n) \cos \tfrac{1}{2} ht,$$
$$B_n = (a_{n-h} + a_n) \cos \tfrac{1}{2} ht - (b_{n-h} - b_n) \sin \tfrac{1}{2} ht.$$

---

* *Compendium der höheren Analysis*, vol. 1, §40.

† *Annales de l'École normale supérieure*, ser. 3, vol. 12 (1895), p. 351.

‡ Annals of Mathematics, vol. 7, nos. 2 and 3, 1906.

Special cases of (β), when $t = \pi/h$ and $t = 0$ respectively, are the convergence of

$$(\beta_1) \qquad \sum_1^\infty (|a_{n-h} - a_n| + |b_{n-h} - b_n|)$$

or

$$(\beta_2) \qquad \sum_1^\infty (|a_{n-h} + a_n| + |b_{n-h} + b_n|).$$

As a particular case under (β_1) may be noted the condition

$$(\beta_3) \qquad a_{n-h} \geqq a_n, \qquad b_{n-h} \geqq b_n; \qquad n \geqq N.$$

This reduces to the theorem of Schlömilch when $h = 1$.

For the proof, let $S_k(x)$ denote the first k terms of (1) and form the following expression in which all a's and b's with negative or zero subscripts are to be put equal to zero:

$$(2) \quad 2\cos\frac{h}{2}(x-t)\,S_k(x)$$

$$= \sum_1^k \left[2a_n \cos nx \cos\frac{h}{2}(x-t) + 2b_n \sin nx \cos\frac{h}{2}(x-t)\right]$$

$$= \sum_1^k \left\{ a_n \left[\cos\left(nx + \frac{h}{2}x - \frac{h}{2}t\right) + \cos\left(nx - \frac{h}{2}x + \frac{h}{2}t\right)\right] \right.$$

$$\left. + b_n \left[\sin\left(nx + \frac{h}{2}x - \frac{h}{2}t\right) + \sin\left(nx - \frac{h}{2}x + \frac{h}{2}t\right)\right]\right\}$$

$$= \sum_{1+h}^{k+h} \left[a_{n-h} \cos\left(nx - \frac{h}{2}x - \frac{h}{2}t\right) + b_{n-h} \sin\left(nx - \frac{h}{2}x - \frac{h}{2}t\right)\right]$$

$$+ \sum_1^k \left[a_n \cos\left(nx - \frac{h}{2}x + \frac{h}{2}t\right) + b_n \sin\left(nx - \frac{h}{2}x + \frac{h}{2}t\right)\right]$$

$$= \sum_1^k \left\{ \left[(a_{n-h} - a_n)\sin\frac{h}{2}t + (b_{n-h} + b_n)\cos\frac{h}{2}t\right]\sin\left(n - \frac{h}{2}\right)x \right.$$

$$\left. + \left[(a_{n-h} + a_n)\cos\frac{h}{2}t - (b_{n-h} - b_n)\sin\frac{h}{2}t\right]\cos\left(n - \frac{h}{2}\right)x \right\}$$

$$+ R_k(x),$$

where

$$R_k(x) = \sum_{k+1}^{k+h}\Big[(a_{n-h}\sin\frac{h}{2}t + b_{n-h}\cos\frac{h}{2}t)\sin(n-\frac{h}{2})x + (a_{n-h}\cos\frac{h}{2}t - b_{n-h}\sin\frac{h}{2}t)\cos(n-\frac{h}{2})x\Big].$$

By use of conditions (α), $\lim R_k(x) = 0$ uniformly, and by (β), the last Σ in (2) converges uniformly: the same is true after dividing by $2\cos\frac{1}{2}h(x-t)$; hence $\lim S_k(x)$, or series (1), converges uniformly, in any interval which does not reach up to a root of the equation $\cos\frac{1}{2}h(x-t) = 0$.

2. Sufficient conditions for establishing the existence of the first derivative of series (1) are obtained in the following theorem, which also provides the means for writing this derivative.

Take h any integer, t a real constant, and let c_1 and c_2 be two points lying between two successive roots of the equation

$$\cos\frac{h}{2}(x-t) = 0.$$

(γ) *Suppose series* (1) *convergent at one point* c, $c_1 \leqq c \leqq c_2$.

(δ) *Assume the uniform convergence of*

$$\sum_1^\infty\Big\{\Big[-\Big((n-h)a_{n-h}+na_n\Big)\cos\frac{h}{2}t+\Big((n-h)b_{n-h}-nb_n\Big)\sin\frac{h}{2}t\Big]\sin(n-\frac{h}{2})x + \Big[\Big((n-h)a_{n-h}-na_n\Big)\sin\frac{h}{2}t+\Big((n-h)b_{n-h}+nb_n\Big)\cos\frac{h}{2}t\Big]\cos(n-\frac{h}{2})x\Big\}$$

in the interval $c_1 \leqq x \leqq c_2$.

(ϵ) *Lastly, let* $\lim a_n = \lim b_n = 0$.

Then in this interval, series (1) *converges uniformly, represents a continuous function $f(x)$, and has a first derivative $f'(x)$ given by dividing series* (δ) *by* $2\cos\frac{1}{2}h(x-t)$.

As before, all coefficients with negative or zero subscripts are to be put equal to zero.

In practice the following special forms, obtained from (δ) when $t = \frac{\pi}{h}$ and $t = 0$ respectively, are more useful for calculating the derivative:

$$(\delta_1) \qquad f'(x) = \frac{1}{2 \sin \frac{1}{2}hx} \sum_1^\infty \left\{ \left[(n-h)a_{n-h} - na_n \right] \cos(n - \frac{h}{2})x + \left[(n-h)b_{n-h} - nb_n \right] \sin(n - \frac{h}{2})x \right\},$$

$$(\delta_2) \qquad f'(x) = \frac{1}{2 \cos \frac{1}{2}hx} \sum_1^\infty \left\{ -\left[(n-h)a_{n-h} + na_n \right] \sin(n - \frac{h}{2})x + \left[(n-h)b_{n-h} + nb_n \right] \cos(n - \frac{h}{2})x \right\}.$$

To obtain this theorem, let $S_k'(x)$ denote the derivative of $S_k(x)$, then

$$(4) \qquad 2\cos\frac{h}{2}(x-t)\,S_k'(x)$$

$$= \sum_1^k \left[-2na_n \sin nx \cos\frac{h}{2}(x-t) + 2nb_n \cos nx \cos\frac{h}{2}(x-t) \right]$$

$$= \sum_1^k \left\{ -na_n \left[\sin(nx + \frac{h}{2}x - \frac{h}{2}t) + \sin(nx - \frac{h}{2}x + \frac{h}{2}t) \right] + nb_n \left[\cos(nx + \frac{h}{2}x - \frac{h}{2}t) + \cos(nx - \frac{h}{2}x + \frac{h}{2}t) \right] \right\}$$

$$= \sum_{1+h}^{k+h} \left[-(n-h)a_{n-h} \sin(nx - \frac{h}{2}x - \frac{h}{2}t) + (n-h)b_{n-h} \cos(nx - \frac{h}{2}x - \frac{h}{2}t) \right] + \sum_1^k \left[-na_n \sin(nx - \frac{h}{2}x + \frac{h}{2}t) + nb_n \cos(nx - \frac{h}{2}x + \frac{h}{2}t) \right]$$

$$= \sum_1^k \left\{ \left[-\left((n-h)a_{n-h} + na_n \right) \cos\frac{h}{2}t + \left((n-h)b_{n-h} - nb_n \right) \sin\frac{h}{2}t \right] \sin(n - \frac{h}{2})x + \left[\left((n-h)a_{n-h} - na_n \right) \sin\frac{h}{2}t + \left((n-h)b_{n-h} + nb_n \right) \cos\frac{h}{2}t \right] \cos(n - \frac{h}{2})x \right\} + R_k(x),$$

where

$$R_k(x) = \sum_{k+1}^{k+h} \left\{ \left[-(n-h)a_{n-h} \cos \frac{h}{2} t + (n-h) b_{n-h} \sin \frac{h}{2} t \right] \sin (n - \frac{h}{2}) x \right.$$
$$\left. + \left[(n-h) a_{n-h} \sin \frac{h}{2} t + (n-h) b_{n-h} \cos \frac{h}{2} t \right] \cos (n - \frac{h}{2}) x \right\}.$$

Let $G_k(x)$ denote the continuous function obtained by dividing each term of the last Σ in (4) by $2 \cos \frac{1}{2} h(x-t)$. Then from (4),

$$(5) \quad S_k(x) = S_k(c) + \int_c^x G_k(x)\, dx + \int_c^x \frac{R_k(x)}{2 \cos \frac{1}{2} h(x-t)}\, dx.$$

We will now show that each of the three terms on the right of (5) approaches a limit uniformly for all values of x as k becomes infinite. This is true of the first term by (γ); call this limit C. By (δ) the second term approaches uniformly $\int_c^x G(x)dx$, if we denote by $G(x)$ the value of the series (δ) divided by $2 \cos \frac{1}{2} h(x-t)$. In the third term, integrate by parts, first replacing $R_k(x)$ by its value, and putting $(n-h)\, A_{n-h}$ and $(n-h)\, B_{n-h}$ in place of the expressions in the brackets. We obtain,

$$\int_c^x \frac{R_k(x)}{2 \cos \frac{1}{2} h(x-t)} dx$$
$$= \sum_{k+1}^{k+h} \int_c^x (n-h) \left[\frac{A_{n-h} \sin (n - \frac{1}{2}h)x + B_{n-h} \cos (n - \frac{1}{2}h)x}{2 \cos \frac{1}{2} h(x-t)} \right] dx$$
$$= \sum_{k+1}^{k+h} \frac{n-h}{n - \frac{1}{2}h} \left\{ \left[\frac{-A_{n-h} \cos (n - \frac{1}{2}h)x + B_{n-h} \sin (n - \frac{1}{2}h)x}{2 \cos \frac{1}{2} h(x-t)} \right]_c^x \right.$$
$$\left. - \frac{h}{2} \int_c^x \left[A_{n-h} \cos (n - \frac{h}{2})x - B_{n-h} \sin (n - \frac{h}{2}) x \right] \frac{\sin \frac{1}{2} h(x-t)}{2 \cos^2 \frac{1}{2} h(x-t)} dx \right\}.$$

Take k so large that $\frac{n-h}{n - \frac{1}{2}h} < 2$, and take K so that $0 < K \leqq \cos^2 \frac{h}{2} (x-t)$; also note that $|x - c| < 2\pi$. Then

$$\left| \int_c^x \frac{R_k(x)}{2 \cos \frac{1}{2} h(x-t)} dx \right| \leqq \frac{2 + \frac{1}{2} h \pi}{K} \sum_{k+1}^{k+h} \left[|A_{n-h}| + |B_{n-h}| \right].$$

Then by (ϵ),

$$\lim_{k=\infty} \int_c^x \frac{R_k(x)}{2\cos\frac{1}{2}h(x-t)}\,dx = 0, \text{ uniformly.}$$

Having thus shown that the three terms on the right hand side of (5) approach limits uniformly, it follows that $S_k(x)$ approaches a limit, which we will call $f(x)$, uniformly, and

$$f(x) = C + \int_c^x G(x)dx,$$

hence,

$$f'(x) = G(x).$$

This is the desired derivative, and gives the equations (δ_1) and (δ_2) for the values of t indicated above.

The usual rule for differentiating the series (1), requiring the uniform convergence of the derived series, is the special case $h = 0$. Lerch's theorem is contained in (δ_1), when $h = 2$.

3. In conclusion, we consider two numerical examples.

Example 1. $$\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots.$$

Conditions (a) and (β_3) are satisfied, for every value of h. The largest interval of convergence is given by the value $h = 1$, whence the series converges uniformly between the roots of $\sin\frac{1}{2}x = 0$.

The derivative is given by (δ_1) where $h = 1$, and is $f'(x) = -\frac{1}{2}$. Then the sum of the series is

$$f(x) = \frac{2k+1}{2}\pi - \frac{x}{2}, \quad 2k\pi < x < (2k+2)\pi, \quad k = 0, 1, 2, \cdots,$$

the constant being obtained by putting $x = (2k+1)\pi$ in the given series.

Example 2. $$\sin x + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots.$$

When $h = 2$, (β_2) becomes

$$\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots < 2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$$

Hence the series converges uniformly between the roots of $\cos x = 0$.

By formula (δ_3), when $h = 2$,

$$f'(x) = \frac{1 + \cos x}{2 \cos x},$$

whence

$$f(x) = \frac{1}{2}\left[x + \log \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) - k\pi - \log \tan \frac{2k+1}{4}\pi\right]$$

in the intervals

$$(2k-1)\frac{\pi}{2} < x < (2k+1)\frac{\pi}{2}, \quad k = 0, \pm 1, \pm 2, \ldots.$$

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JUNE, 1906.

NOTE ON THE DEFINITION OF AN ABELIAN GROUP BY INDEPENDENT POSTULATES

By W. A. Hurwitz

E. V. Huntington has given two definitions of an Abelian group by sets of three and four independent postulates respectively.* I propose two definitions by sets of two and three independent postulates. These definitions are explicitly modelled after those of Huntington; they differ from the latter, however, in that the associative and commutative laws are replaced by a single law, logically as simple as either, and furnishing (in connection with the other postulates of each set) a necessary and sufficient condition for the subsistence of both the associative and commutative laws.

For convenience of reference, I shall re-state Huntington's first definition:—

A system made up of a set of elements, $a, b, c, \cdots$, with a rule of combination, $\circ$, is called an Abelian group when the following conditions are satisfied:†

$H1$). $a \circ b = b \circ a$, *whenever* a, b *and* $b \circ a$ *belong to the set.*

$H2$). $(a \circ b) \circ c = a \circ (b \circ c)$, *whenever* a, b, c, $a \circ b$, $b \circ c$ *and* $a \circ (b \circ c)$ *belong to the set.*

$H3$). *For every two elements* a *and* b *there is an element* x *in the set such that* $a \circ x = b$.

The distinction between finite Abelian groups of various orders and infinite Abelian groups is made by adding:—

a). *The set contains* n *elements;* or

b). *The set is infinite.*

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* *Transactions of the American Mathematical Society*, vol. 4 (1903), pp. 27-30. For bibliographical references to other definitions, see *ibid*, vol. 6 (1905), p. 181.

† While the usage in regard to the words "set" and "system" is not wholly uniform, it seems desirable to use "set" or "class" for any collection of objects regarded simply as distinguishable entities, and "system" for a collection of objects *plus* some relations or rules of combination.

1. Definition by two postulates. I define an Abelian group by the following postulates :—

1). $(a \circ b) \circ c = a \circ (c \circ b)$, *whenever* $a, b, c, a \circ b, c \circ b$ *and* $(a \circ b) \circ c$ *belong to the set.*

2). *For every two elements* a *and* b *there is an element* x *in the set such that* $a \circ x = b$.

Postulate 2) is precisely $H3$). Thus, to justify the characterization of the sytem just defined as an Abelian group, it will suffice to deduce $H1$), $H2$) from 1), 2).

Proof of $H1$). Suppose a, b and $b \circ a$ belong to the set. By 2), take x so that $a \circ x = b$. Then by 1), since a, x, $a \circ x$, and $(a \circ x) \circ a$ belong to the set,

$$(a \circ x) \circ a = a \circ (a \circ x) ;$$

that is,

$$b \circ a = a \circ b.$$

Proof of $H2$). Suppose a, b, c, $a \circ b$, $b \circ c$, and $a \circ (b \circ c)$ belong to the set. Then by $H1$),

$$a \circ (b \circ c) = (b \circ c) \circ a = (c \circ b) \circ a,$$

and by 1),

$$(c \circ b) \circ a = c \circ (a \circ b).$$

Again, by $H1$),

$$c \circ (a \circ b) = (a \circ b) \circ c ;$$

so that

$$a \circ (b \circ c) = (a \circ b) \circ c.$$

Therefore an Abelian group is defined by 1), 2).

Independence of 1), 2), *and* a).

The postulates 1), 2), a) are independent when $n > 1$. The proof is established in the usual way by the following systems :

(1). The first n positive integers, with $a \circ b = b$; or the n distinct integers modulo n, with $a \circ b \equiv -a - b \pmod{n}$. The former example is associative but not commutative, the latter is commutative but not associative.

(2). The first n positive integers, with $a \circ b = 1$.

(a). Any infinite Abelian group.

Independence of 1), 2), *and* b).

Consider in the same manner, the following systems:

(1). All positive integers, with $a \circ b = b$; or again, all rationals, with $a \circ b = 2(a + b)$.

(2). All positive integers, with $a \circ b = a + b$.

(b). Any finite Abelian group.

2. Definition by three postulates. The set of postulates modelled after Huntington's second set is as follows:

1'). $(a \circ b) \circ c = a \circ (c \circ b)$, *whenever* a, b, c, $a \circ b$, $c \circ b$, $(a \circ b) \circ c$, *and* $a \circ (c \circ b)$ *belong to the set.*

2'). *For every two elements* a *and* b, *there is an element* x' *in the set such that* $(a \circ x') \circ b = b$.

3'). *If* a *and* b *belong to the set, then* $a \circ b$ *belongs to the set.*

That 1'), 2'), 3') really define an Abelian group is made clear by noting that from them may be deduced 2), by taking $x = b \circ x'$.

Independence of 1'), 2'), 3'), *and* a).

The non-group systems (1'), (2'), (a) are the same as (1), (2), (a), used previously.

For (3'), take the first n positive integers, with $a \circ b = 1$, unless $a = 1$ or $b = 1$; $1 \circ b = b$; $a \circ 1$ not in the set unless $a = 1$. Here 2') is satisfied when $x' = a$.

Independence of 1'), 2'), 3'), *and* b).

The non-group systems (1'), (2'), (b) are the same as (1), (2), (b), used previously.

For (3'), take all integers with $a \circ b = a + b$ when a or b or $a + b$ is zero; otherwise $a \circ b$ not in the set.

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APRIL, 1906.

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